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BREAKING OF WAVES OF LIMITING AMPLITUDE OVER AN OBSTACLE

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Homogeneous heavy fluid flows over an uneven bottom are studied in a long-wave approximation. A mathematical model is proposed that takes into account both the dispersion effects and the formation of a turbulent upper layer due to the breaking of surface gravity waves. The asymptotic behavior of nonlinear perturbations at the wave front is studied, and the conditions of transition from smooth flows to breaking waves are obtained for steady-state supercritical flow over a local obstacle.

Key words: *homogeneous fluid, supercritical flow, waves of limiting amplitude, wave breaking.*

Introduction. Wave processes in free-surface homogeneous heavy fluid flows are studied using mathematical models corresponding to the second approximation of shallow-water theory (versions of the Boussinesq equations [1], Green–Naghdi equations [2], Zheleznyak–Pelinovskii equations [3], etc.). These equations adequately describe the structure of nonlinear wave fronts of moderate amplitude. However, unlike in the accurate formulation of the Cauchy–Poisson problem, in the second approximation, it is impossible to describe waves of limiting amplitude and obtain a criterion for transition from smooth to breaking waves. The breaking of surface waves has been actively studied experimentally in the last decade [4–6]. It has been shown that the breaking gives rise to a near-surface turbulent layer, which plays an important role in the formation of the wave front. Theoretical models of this process were constructed only for developed turbulent bores [7–9].

In the present paper, we study a mathematical model that takes into account the effect of the surface turbulent layer on the structure of steady flow in the neighborhood of the local obstacle.

1. Mathematical Model. The shallow water equations for an incompressible fluid taking into account the surface turbulent layer and the nonhydrostatic nature of the pressure distribution can be written as follows (see [10, chapter 6]):

$$
\begin{gathered}
h_t + (hu)_x = -\sigma q,\\
u_t + uu_x + g(h+\eta+z)_x + p_x = 0,\\
\eta_t + (\eta v)_x = \sigma q, \qquad (1)\\
v_t + vv_x + g(h+\eta+z)_x = \sigma q(u-v)/\eta,\\
q_t + vq_x = \sigma\Big((u-v)^2 - (1+\theta)q^2\Big)/(2\eta).
\end{gathered}
$$

Here h and η are the depths of the lower (potential) and upper (turbulent) layers, u and v are the corresponding mean horizontal velocities in the layers, q is the root mean square velocity of small-scale motion in the upper layer, g is the acceleration of gravity, $z = z(x)$ is the shape of the bottom, and the coefficients σ and θ are constant.

To close the model, it is necessary to choose an expression for the additional pressure p at the channel bottom due to nonhydrostatic effects. For $p \equiv 0$, Eqs. (1) are the first approximation of shallow water theory taking

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into account of surface-wave breaking. The structure of traveling waves for this model was studied in [9, 10]. For $p = p(h, \dot{h}, \ddot{h})$ and $\dot{h} = \partial/\partial t + u\,\partial/\partial x$, we obtain a generalization of the various models corresponding to the second approximation. Free-surface steady flows in the neighborhood of a local obstacle are described using the following representation for the excess pressure:

$$p = u^2(2hh_{xx} - h_x^2 + 3k(z_x^2 + hz_{xx}))/6. \tag{2}$$

For $k = 1$, relation (2) is obtained in [11], and for $k = 0$, in [12]. For steady-state flows, system (1), (2) becomes

$$(hu)_x = -\sigma q, \qquad (\eta v)_x = \sigma q,$$

$$uu_x + g(h + \eta + z)_x + \Big(u^2(2hh_{xx} - h_x^2 + 3k(z_x^2 + hz_{xx}))\Big)_x/6 = 0, \tag{3}$$

$$vv_x + g(h + \eta + z)_x = \sigma q(u - v)/\eta, \qquad vq_x = \sigma((u - v)^2 - (1 + \theta)q^2)/(2\eta).$$

We consider smooth solutions (3) that describe steady-state perturbations of uniform supercritical flow ($h = h_0$, $u = u_0$, and $\mathrm{Fr} = u_0/\sqrt{gh_0} > 1$) past a local symmetric obstacle

$$z(x) = z(-x), \qquad z(x) = 0 \quad \text{at} \quad |x| > l. \tag{4}$$

It is assumed that at a large enough distance from the obstacle, the upstream flow is not perturbed, i.e.,

$$h \to h_0, \quad u \to u_0, \quad \eta \to 0 \qquad \text{at} \quad x \to -\infty. \tag{5}$$

It is required to find the possible wave configurations localized in the neighborhood of the obstacle. We first solve this problem using a simpler model assuming that a surface turbulent layer is absent ($\eta \equiv 0$).

2. Solitary Waves over an Obstacle. For $\eta \equiv 0$, system (3) reduces to the model obtained in [11]:

$$hu = Q \equiv \text{const},$$
$$u^2/2 + g(h + z) + u^2(2hh_{xx} - h_x^2 + 3k(z_x^2 + hz_{xx}))/6 = J = \text{const}. \tag{6}$$

In [11], Eqs. (6) were used to construct transcritical flow regimes over an obstacle ($\mathrm{Fr} < 1$).

For $\mathrm{Fr} > 1$, the obstacles located in the neighborhood of the flow are specified by the solutions of Eqs. (6) that satisfy conditions (5) and are symmetric about the coordinate origin. Such solutions can be obtained by perturbations of two types of flows over an even bottom: uniform flow and a solitary wave. Therefore, in a certain range of Froude numbers $\mathrm{Fr} > 1$ in the neighborhood of the obstacle, two different flows can occur.

Since the flow outside the obstacle (at $|x| > l$) is part of a soliton whose shape can be obtained by integrating the equations for $z = 0$, relation (6) reduces to one first-order equation

$$h_x^2 = \frac{3}{\mathrm{Fr}^2 h_0^3}\,(h - h_0)^2(\mathrm{Fr}^2 h_0 - h), \tag{7}$$

whose solution is represented in quadratures. We note that in (7) conditions (5) are used, i.e,

$$Q = h_0 u_0, \qquad J = (\mathrm{Fr}^2/2 + 1)gh_0.$$

Thus, the problem of seeking symmetric flows in the neighborhood of a local obstacle reduces to constructing solutions (6) in the interval $(-l, 0)$ with the boundary conditions

$$h_x\Big|_{x=-l} = \Big(\frac{3(\mathrm{Fr}^2 h_0 - h)}{\mathrm{Fr}^2 h_0^3}\Big)^{1/2}(h - h_0), \qquad h_x\Big|_{x=0} = 0. \tag{8}$$

The solution of problem (6), (8) can be found numerically for a specified shape of the obstacle. For smooth obstacles of simple shapes, it can be shown that there is a critical value $\mathrm{Fr}_* > 1$ such that for $\mathrm{Fr} > \mathrm{Fr}_*$, problem (6), (8) has two solutions, and with increasing Froude number, the wave amplitude corresponding to the perturbed soliton increases without bound [13, 14].

Figure 1 shows symmetric wave profiles for supercritical flow past a local obstacle (semicylinder) obtained by solitary-wave perturbation (curve 1) and by uniform-flow perturbation (curve 2) using model (6) for $k = 0$. In contrast to the exact Cauchy–Poisson formulation [13], various modifications of the shallow water equations over an uneven bottom (second approximation), including the Korteweg–de Vries equations [14, 15], do not have constraints

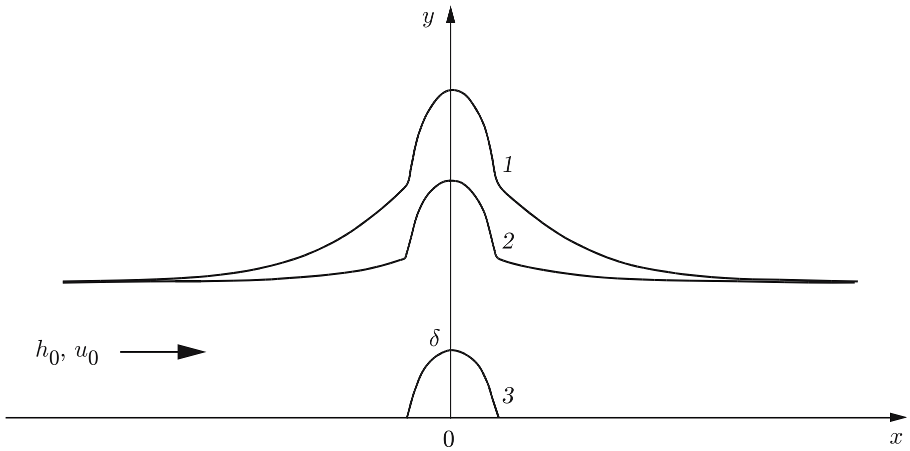

Fig. 1. Supercritical flow past an obstacle [model (6), Fr = 1.5, $\sigma = 0.15$, $\delta = 0.5h_0$, and $k = 0$]: 1) solitary-wave perturbation; 2) uniform-flow perturbation; 3) obstacle (a semicylinder of radius $R = 0.5h_0$).

on the wave amplitude and, hence, do not give a criterion for transition from smooth to breaking waves. In Sec. 3, it was shown that model (3), which takes into account the formation of a surface turbulent layer, allows not only to determine waves of limiting amplitude but also to describe the process of wave breaking and development of a turbulent bore.

We note that for short obstacles, Eqs. (6) lose its meaning. In this case, problem (6), (8) can be solved in the class of continuous piecewise smooth functions if the concentrated effect of the obstacle on long-wave motion is added to the equations [16]. This implies that the steady-state supercritical flow in the neighborhood of an obstacle of small length is a certain part of a soliton at $-\infty < x < 0$ and $0 < x < \infty$ with a specified slope angle $h_x\Big|_{x=0-0} = -h_x\Big|_{x=0+0}$ dependent on the shape of the obstacle. As a first approximation, this angle can be specified as follows [16]:

$$h_x\Big|_{x=0-0} = \frac{3}{2}\,\frac{gh_0}{Q^2}\int\limits_{-l}^{l} z(x)\,dx. \tag{9}$$

Problem (6), (9) is much simpler than problem (6), (8) considered above, and its solution can be found in quadratures.

3. Structure of the Wave Bore in a Horizontal Channel. Equations (3) describing nonlinear waves in a homogeneous fluid taking into account the development of a surface turbulent layer are much more complex than Eqs. (6). Nevertheless, for a channel with an even bottom [$z(x) \equiv 0$], we can construct solution (3), which is an analog of the soliton. We consider the nontrivial solution (3), which is formed from uniform flow, i.e.,

$$h \to h_0, \quad \eta \to 0, \quad u \to u_0 \qquad \text{at} \quad x \to -\infty. \tag{10}$$

We note that the right side of Eqs. (3) is singular for $\eta \to 0$. Therefore, for small perturbations $\tilde h$, $\tilde u$, $\tilde\eta$, $\tilde v$, and $\tilde q$ of the uniform flow

$$h = h_0 + \tilde h, \quad \eta = \tilde\eta, \quad u = u_0 + \tilde u, \quad v = u_0 + \tilde v, \quad q = \tilde q,$$

we obtain a semilinear system which is homogeneous in perturbations:

$$\begin{gathered} u_0\tilde h_x + h_0\tilde u_x = -\sigma\tilde q, \qquad u_0\tilde\eta_x = \sigma\tilde q, \\ g(\tilde h_x + \tilde\eta_x) + u_0\tilde u_x + u_0^2 h_0\tilde h_{xxx}/3 = 0, \qquad g(\tilde h_x + \tilde\eta_x) + u_0\tilde v_x = \sigma\tilde q(\tilde u - \tilde v)/\tilde\eta, \\ u_0\tilde q_x = \sigma((\tilde u - \tilde v)^2 - (1+\theta)\tilde q^2)/(2\tilde\eta). \end{gathered} \tag{11}$$

Solutions (11) subject to conditions (10) can be written as $\tilde h = \hat h\,e^{\lambda x}$, $\tilde\eta = \hat\eta\,e^{\lambda x}$, $\tilde u = \hat u\,e^{\lambda x}$, $\tilde v = \hat v\,e^{\lambda x}$, and $\tilde q = \hat q\,e^{\lambda x}$ with a positive parameter λ satisfying the following algebraic system of equations:

$$\lambda u_0\hat h + \lambda h_0 \hat u = -\sigma \hat q, \qquad \lambda u_0 \hat\eta = \sigma \hat q,$$
$$g\hat h + g\hat\eta + u_0\hat u + \lambda^2 u_0^2 h_0 \hat h/3 = 0, \qquad \lambda(g\hat h + g\hat \eta + u_0 \hat v) = \sigma \hat q(\hat v - \hat u)/\hat\eta, \tag{12}$$
$$\lambda u_0 \hat q = -\sigma((1+\theta)\hat q^2 - (\hat u - \hat v)^2)/(2\hat\eta).$$

It is required to find a nontrivial solution of (12) with the additional constraints $\hat\eta > 0$, $\hat q > 0$, and $\lambda > 0$. The first four equations (12) imply the relations

$$\hat u - \hat v = \frac{(gh_0 - u_0^2)(\hat h + \hat\eta)}{2u_0 h_0}, \qquad \hat h = \frac{3(u_0^2 - gh_0)\hat\eta}{h_0^2 u_0^2 \lambda^2 + 3(gh_0 - u_0^2)},$$
$$\hat h + \hat\eta = \frac{h_0^2 u_0^2 \lambda^2 \hat\eta}{h_0^2 u_0^2 \lambda^2 + 3(gh_0 - u_0^2)}. \tag{13}$$

A consequence of (13) is

$$(\hat u - \hat v)^2 = \frac{(gh_0 - u_0^2)^2 h_0^2 u_0^2 \lambda^4 \hat\eta^2}{4(h_0^2 u_0^2 \lambda^2 + 3(gh_0 - u_0^2))^2}. \tag{14}$$

From the second and fifth equations (12), we obtain

$$(\hat u - \hat v)^2 = (3+\theta)\hat q^2 = (3+\theta)\lambda^2 u_0^2 \hat\eta^2/\sigma^2. \tag{15}$$

From (14) and (15), we infer the following quadratic equation for the values of the parameter λ ($\lambda > 0$) for which there is a nonzero solution of (12):

$$h_0^2 u_0^2 \lambda^2 - p h_0 (u_0^2 - gh_0)\lambda + 3(gh_0 - u_0^2) = 0. \tag{16}$$

Here $p = \pm\sigma/(2\sqrt{3+\theta}\,)$. For supercritical flows (Fr > 1), the positive roots of Eqs. (17) are specified by the formula

$$\lambda = \frac{p h_0(u_0^2 - gh_0) + \sqrt{p^2 h_0^2 (u_0^2 - gh_0)^2 + 12 h_0^2 u_0^2 (u_0^2 - gh_0)}}{2h_0^2 u_0^2}. \tag{17}$$

In (17), an analog of the soliton is constructed using the positive values of p, and for $\sigma \to 0$ ($p \to 0$), the values of λ coincide with the corresponding values for the soliton obtained using (7):

$$\lambda = \sqrt{3(\mathrm{Fr}^2 - 1)}\Big/(\mathrm{Fr}\, h_0).$$

For the specified Froude number (Fr > 1), the steady-state solution (3) that satisfies (10) can be constructed using the asymptotics obtained. For near-critical flows ($0 < \mathrm{Fr} - 1 \ll 1$), the structure of solution (3) with asymptotics (10), (17) is close to the soliton specified by (7) since the relative thickness of the upper turbulent layer is smaller than the total depth of the flow. Nevertheless, the solution obtained is no longer symmetric and is a wave bore with a turbulent layer of gradually increasing thickness (see [10, chapter 6]).

The main feature that distinguishes the model considered from (6) is that the waves described by system (3) reach the limiting amplitude. This implies that there is a critical value of the Froude number $\mathrm{Fr}^* > 1$ such that a smooth solution of (3) with asymptotics (10) does not exist for $\mathrm{Fr} > \mathrm{Fr}^*$. To understand the reason for the breaking of the smooth solutions, we consider system (3) written as

$$h_x = w, \qquad u_x = -\frac{\sigma q + uw}{h},$$
$$w_x = \frac{1}{2h}\Big(\frac{6}{u^2}\Big(E - g(h+\eta+z) - \frac{u^2}{2}\Big) + w^2 - 3k(z_x^2 + h z_{xx})\Big), \tag{18}$$
$$\eta_x = \frac{g\eta(w + z_x) + \sigma q(2v - u)}{\Delta}, \qquad v_x = \frac{\sigma q - v\eta_x}{\eta}, \qquad q_x = \frac{\sigma}{2\eta v}\Big((u-v)^2 - (1+\theta)q^2\Big),$$

where $\Delta = v^2 - g\eta$ and $E = u_0^2/2 + gh_0$.

We note that for $x \to -\infty$, the solution of problem (18), (10) tends exponentially to uniform flow. In particular, $v = \hat v\, \mathrm{e}^{-\lambda x}$ and $\eta = \hat\eta\, \mathrm{e}^{-\lambda x}$; therefore, $\Delta = v^2 - g\eta < 0$ as $x \to -\infty$. If the condition $\Delta < 0$ is valid for the entire flow region up to the point the upper turbulent layer reaches the channel bottom ($h = 0$), solution (18) over an even bottom ($z = 0$) is smooth and is a wave bore. However, in the case where the function Δ in (3) vanishes at the wave front, transition from subcritical flow ($\Delta < 0$) to supercritical flow ($\Delta > 0$) with respect to the wave mode generated by the upper turbulent layer can occur by means of a hydraulic jump. The appearance of a hydraulic jump followed by a region of intense turbulent mixing in the solution corresponds to the development of a roller at the leading edge of the breaking wave. In the present study, discontinuous solutions (3) are not considered; therefore, as the criterion of transition to breaking waves we use the vanishing of the quantity Δ in the solutions of problem (3), (10). For the wave bore, the critical value of the Froude number Fr^* for which the value of Δ vanishes at the front of the first wave can be found numerically. For an even bottom, $\mathrm{Fr}^* \simeq 1.3$ and solution (18), (10) gives the following value of the limiting-wave amplitude:

$$(h + \eta)_{\max} = 1.9 h_0 \quad \text{at} \quad \mathrm{Fr}^* = 1.3. \tag{19}$$

The values obtained in (19) agree with the parameters of solitary waves of limiting amplitude found from the Cauchy–Poisson problem in the exact formulation [13].

4. Supercritical Flow over a Local Obstacle. We again consider steady-state flow over a smooth obstacle (4) of maximum height $\delta = z(0)$ which is symmetric about the coordinate origin and is located in the interval $(-l, l)$. As for Eqs. (6), in the neighborhood of the obstacle, two types of supercritical flows can occur: 1) flows obtained by perturbation of the uniform flow ($h \equiv h_0$, $u \equiv u_0$, and $\eta = 0$); 2) steady-state flows describing a wave bore. In this case, for $x < -l$, the steady-state solution of problem (3), (10) is specified by the one-parameter family $h = h_1(x - x_s)$, $u = u_1(x - x_s)$, $\eta = \eta_1(x - x_s)$, $v = v_1(x - x_s)$, $q = q_1(x - x_s)$, where the point x_s defines the position of the wave front relative to the obstacle. To construct the wave profile over the obstacle, it suffices to solve the Cauchy problem for (18) with the following initial data for $x = -l$:

$$\begin{gathered} h(-l) = h_1(-l - x_s), \qquad u(-l) = u_1(-l - x_s), \qquad \eta(-l) = \eta_1(-l - x_s), \\ v(-l) = v_1(-l - x_s), \qquad q(-l) = q_1(-l - x_s). \end{gathered} \tag{20}$$

Unlike in the case considered in Sec. 2, the solution of problem (18), (20) is not symmetric with respect to the coordinate origin because of the development of a surface turbulent layer. Nevertheless, an analog of the soliton over an obstacle can also be constructed for system (18) as a special solution that separates two types of flow: flows with lee waves and flows with a gradient catastrophe.

For a small obstacle ($\delta \ll h_0$), the structure of the wave bore changes insignificantly compared to the corresponding solution for a horizontal channel. For an obstacle comparable to the fluid depth ($\delta \sim h_0$), by virtue of the nonlinearity of the system for some values of x_s, the solution of problem (18), (20) can fail on a finite segment of the channel due to an unbounded increase in the derivatives (a gradient catastrophe). However, with a corresponding choice of the parameter x_s, a flow regime with lee waves can be implemented. Therefore, a certain range of parameters Fr and δ contains a critical value of x_s^* that separates different flow regimes. Exactly this limiting flow regime is an analog of the localized soliton-like perturbation over an obstacle considered in Sec. 2. Numerical calculations demonstrate that this flow regime exists in a broad enough range of the determining parameters $\mathrm{Fr} > 1$ and $\delta > 0$.

We note that the localized flow over an obstacle can also be constructed for a fixed position of the wave front x_s by varying the height of the obstacle δ.

Figure 2 shows wave profiles (the solid curves refer to the free surface and the dashed curves refer to the boundary of the turbulent layer) for supercritical flow past a segment of a cylinder calculated using model (3) for $k = 0$. A small change in the height of the obstacle for the specified position of the wave front gives different solutions of problem (18), (20): lee waves (curves 1), perturbed solitary wave (curves 2), and a solution with a gradient catastrophe (curves 3). An analog of the symmetric flow found for model (6) without a turbulent layer (curve 1 in Fig. 1) is the flow described by curve 2.

As noted above, one of the reasons for the breaking of smooth solutions over an even bottom is that the quantity Δ vanishes. The placement of a local obstacle in the flow provides smooth solutions of larger amplitude than that in a channel with an even bottom. Nevertheless, like in a horizontal channel, for supercritical flow past a local obstacle there is a critical value of the Froude number Fr^* such that for $\mathrm{Fr} > \mathrm{Fr}^*$ the solution should contain a hydraulic jump, which in the given model corresponds to wave breaking with the formation of a roll at the wave crest. If the criterion of wave breaking is the vanishing of the function Δ at the leading edge of the local flow

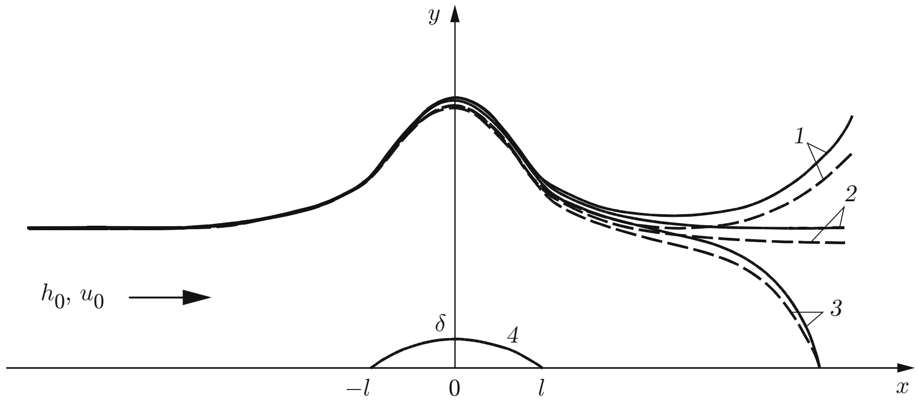

Fig. 2. Supercritical flow past an obstacle [model (3), Fr = 1.4, $\sigma = 0.15$, $\theta = 2$, and $k = 0$]: the solid curves show the free surface; the dashed curves show the lower boundary of the turbulent layer; 1) lee waves ($\delta = 0.19h_0$); 2) solitary-wave perturbation ($\delta = 0.2h_0$); 3) gradient catastrophe ($\delta = 0.21h_0$); 4) obstacle (a segment of a cylinder of radius $R = 10h_0$).

perturbation by an obstacle (curve 2 in Fig. 2), the critical value of Fr^* for this obstacle can be found numerically. If the length of the obstacle l is smaller than the channel depth h_0, the fluid entrainment in the upper turbulent layer over the obstacle can be ignored. In this case, $\sigma = 0$ and Eqs. (3) can be reduced to one equation for the depth h after elimination of the variables η, v, and u from the relations

$$\eta v = \eta_1 v_1 = Q^+, \qquad hu = h_1 u_1 = Q^-,$$
$$v^2/2 + g(h + \eta + z) = v_1^2/2 + g(h_1 + \eta_1) = J^+. \tag{21}$$

We note that

$$\eta_x = g\eta(h_x + z_x)/\Delta, \tag{22}$$

and, by virtue of (21) and (22), the solution of system (3) for $\sigma = 0$ which is symmetric about the coordinate origin can be constructed similarly to the solution of problem (7), (8) for the case where Δ does not vanish. This solution approximately describes the solitary wave profile over a local obstacle. The structure of the solution of (3) is even more simplified by taking into account the effect of an obstacle of small length and using the hypothesis on the concentrated effect of a local change in the bottom shape on the flow. In this case, the solution is constructed in the class of piecewise smooth functions similarly to (9). Because $l \ll h_0$, the boundary conditions are specified as

$$h_x\Big|_{x=0-0} = -h_x\Big|_{x=0+0} = f(h, A),$$

where f is a specified function and A is the cross-sectional area of the obstacle. Thus, the wave profile transforming the flow from supercritical to subcritical in the neighborhood of a local change in the channel depth is obtained by joining the solutions of (3) over an even bottom with the corresponding asymptotic solution at large distances from the obstacle. We also note that in each of the above-listed approximations, waves of limiting amplitude are obtained from the condition of vanishing of the function Δ at any point of the wave front.

Conclusions. The existence of waves of limiting amplitude in steady-state heavy fluid flows in the exact formulation (Cauchy–Poisson problem) is a consequence of the Bernoulli integral applied to the surface layer [13]. For various modifications of the shallow-water equations, this approach is inapplicable by virtue of the corresponding hypotheses on the distribution of the horizontal flow velocity. The inclusion of a thin surface layer in the model allows a mathematical description of the breaking of smooth waves of finite amplitude for a wide class of shallow water equations (the first and second approximations). The following step of this study consists of using Eqs. (3) to construct a mathematical model for breaking waves, in particular, for determining the inner structure of the flow in a turbulent bore.

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